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Comment on a Paper by G. H. Weiss, S. Havlin, and A. Bunde

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A simple proof is pointed out for the asymptotic exponential decay of the n-step survival probability of a random walk on a finite lattice with traps in the limit as $n \to \infty$. Some bounds are mentioned, which are valid for finite n and for symmetric random walks.

In the preceeding paper Weiss, Havlin, and Bunde[1] consider the survival of a random walk on a finite lattice, with periodic boundary conditions, containing a trap. The initial position of the random walk is uniformly distributed on the lattice. One of their results is the *asymptotic exponential* decay of the (average) n-step survival probability, which is denoted by $\langle U_n \rangle$, in the limit as $n \to \infty$. They further show that the exponent is the root of an equation involving the Green's function at the origin. At the end of their paper they write that they expect qualitatively similar results to hold also for lattices with more than one trap and with different boundary conditions. It is the aim of this short note to point out an elementary proof for the latter, to slightly refine the conclusions of the paper, and to mention a few useful bounds for $\langle U_n \rangle$, valid for finite n, for symmetric random walks.

Consider a *finite* set S of s points of which $s-u$ are traps $(0<u<s)$, and a random walk (Markov chain) on S with stepping probabilities $p(l \to l')$, $l, l' \in S$. Let $U \subset S$ be the set of nontrapping points and $\mathbf{p} = [p(l \to l')]_{l,l' \in U}$ the $u \times u$ matrix of stepping probabilities in U. S, U, and $\mathbf{p}$ will be arbitrary. All that will be required to assume is that trapping is certain, i.e., $\langle U_n \rangle \to 0$ as $n \to \infty$. This will amount to the assumption that

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from each point of U there is a walk of positive probability leading to a trap.

The probability $\langle U_n \rangle$ may be written as

$$U_n = \frac{1}{s}(\mathbf{e}, \mathbf{p}^n\mathbf{e}) \tag{1}$$

where $\mathbf{e}$ is the u-vector with all elements equal to 1. Clearly, $\sum_{l' \in U} p(l \to l') \leqslant 1$ for all $l \in U$. Let $\mathbf{p}$ be irreducible. Because $\langle U_n \rangle \to 0$, strict inequality must hold for at least one $l \in U$. From the Perron–Frobenius theorem[2] it then follows that $\mathbf{p}$ has a real eigenvalue $0 < \lambda_1 < 1$ with the following properties: (i) $|\lambda| \leqslant \lambda_1$ for any other eigenvalue λ of $\mathbf{p}$, (ii) with λ_1 are associated strictly positive left and right eigenvectors $\mathbf{x}_1$ and $\mathbf{y}_1$ [which may be chosen such that $(\mathbf{x}_1, \mathbf{y}_1) = 1$]. The fact that $\lambda_1 < 1$ follows through (ii). When $\mathbf{p}$ is reducible, it may, through a permutation of its rows and columns, be written in the form

$$\begin{bmatrix} \mathbf{p}_{11} & 0 \\ \mathbf{p}_{21} & \mathbf{p}_{22} \end{bmatrix}$$

where $\mathbf{p}_{11}$ and $\mathbf{p}_{22}$ are square matrices and $\mathbf{p}_{11}$ is either zero or irreducible. A repetition of the argument shows that again the largest eigenvalue λ_1 is <1 since $\langle U_n \rangle \to 0$, but now λ_1 may be $=0$, in which case $\mathbf{p}$ is nilpotent.

There are two possibilities for the asymptotic behaviour of $\langle U_n \rangle$:

$$\text{(I)} \quad \lambda_1 = 0: \qquad \exists n' \qquad \text{such that } \langle U_n \rangle = 0 \text{ for } n \geqslant n' \tag{2}$$

$$\text{(II)} \quad \lambda_1 > 0: \qquad \lim_{n \to \infty} \frac{1}{n} \log \langle U_n \rangle = \log \lambda_1 \tag{3}$$

The proof is elementary. When $\mathbf{p}$ is irreducible it is known (see Ref. 2, p. 123) that, if m is the period of $\mathbf{p}$, i.e., the number of eigenvalues with modulus λ_1,

$$\lim_{n \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} \left(\frac{\mathbf{p}}{\lambda_1}\right)^{n-k} = \mathbf{x}_1 \mathbf{y}_1 \tag{4}$$

($\mathbf{p}$ is aperiodic if $m = 1$). From (1), (4) and the monotonicity of $\langle U_n \rangle$, (3) immediately follows, for any m. When $\mathbf{p}$ is reducible, either $\mathbf{p}$ is nilpotent and (2) holds, or the largest eigenvalue $\lambda_1 > 0$ of some "irreducible block" dominates so that again (3) holds.

(I) occurs if each point of U has the property that the walk cannot return without hitting a trap. An example is the one-sided nearest-neighbour random walk on a ring of s points with one trap, where $n' = s-1$. (II) occurs whenever U contains at least one point to which return is possible. An example is the symmetric nearest-neighbour random walk (Pólya, or simple random walk) on the same ring; in this case $\lambda_1 = \cos(\pi/s)$ (see Ref. 3, p. 239). The *existence* of the limit in (3) is necessary for equation (14) in Ref. 1 to *have* a root.

When $\mathbf{p}$ is *symmetric* [i.e., $p(l \to l') = p(l' \to l)$ on U], λ_1 has, by Fischer's theorem,[4] the following representation:

$$\lambda_1 = \sup_{(\mathbf{x},\mathbf{x})=1} (\mathbf{x}, \mathbf{px}) \tag{5}$$

Diagonalization of (1) gives

$$\langle U_n \rangle = \frac{1}{s} \sum_{i=1}^{u} c_i^2 \lambda_i^n \tag{6}$$

where λ_i are eigenvalues of $\mathbf{p}$ and c_i are constants with $\sum_i c_i^2 = (\mathbf{e}, \mathbf{e}) = u$. The following bounds are useful:

$$\langle U_n \rangle \leqslant \lambda_1^n \langle U_0 \rangle \tag{7}$$

$$\langle U_n \rangle \geqslant \left\{ \frac{\langle U_1 \rangle}{\langle U_0 \rangle} \right\}^n \langle U_0 \rangle \tag{8}$$

$$\langle U_{n_2} \rangle \langle U_{n_3} \rangle \leqslant \langle U_{n_1} \rangle \langle U_{n_4} \rangle \tag{9}$$

for $n_1 + n_4 = n_2 + n_3$, n_1 and n_4 even, and $n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4$.

(7) follows because $|\lambda_i| \leqslant \lambda_1$ for all i and $\langle U_0 \rangle = u/s$. For a proof of (8) see Ref. 5. For n even (8) is easily seen to follow from Jensen's inequality,[4] since $\sum_i c_i^2 \lambda_i^n / \sum_i c_i^2 \geqslant \{\sum_i c_i^2 \lambda_i\}^n$ by convexity. For n odd, however, a stronger argument is needed. For a proof of (9) see Ref. 6. (9) is shown by writing

$$\langle U_{n_1} \rangle \langle U_{n_4} \rangle - \langle U_{n_2} \rangle \langle U_{n_3} \rangle$$
$$= \frac{1}{2s^2} \sum_{i,j} c_i^2 c_j^2 (\lambda_i \lambda_j)^{n_1} (\lambda_i^{n_2 - n_1} - \lambda_j^{n_2 - n_1})(\lambda_i^{n_3 - n_1} - \lambda_j^{n_3 - n_1}) \tag{10}$$

which is a sum of nonnegative terms. Note that $\langle U_1 \rangle / \langle U_0 \rangle = (\mathbf{e}, \mathbf{pe})/(\mathbf{e}, \mathbf{e})$, which should be compared with (5). $\langle U_n \rangle / \langle U_{n-1} \rangle$ is *not* always monotonic in n.

When $\mathbf{p}$ is *asymmetric* none of these bounds holds generally. The representation of λ_1 is in this case slightly more involved (see Ref. 4, p. 82).

All the results mentioned follow from standard matrix algebra. Only the *finite* size of S is relevant. When this is dropped the asymptotic decay of $\langle U_n \rangle$ is in general *not* exponential (the limits $s \to \infty$ and $n \to \infty$ may not be interchanged).

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